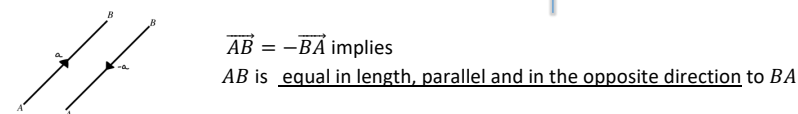
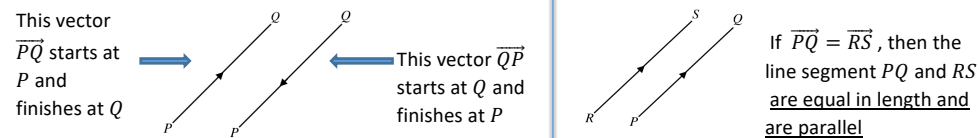


Vectors Cheat Sheet

Vectors

A vector has both **magnitude** and **direction**.

You can represent a vector using a **directed line segment**. Direction of arrow shows the direction of vector



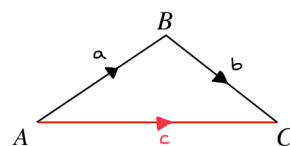
Triangle law for vector addition:

You can add two vector using the triangle law for vector addition.

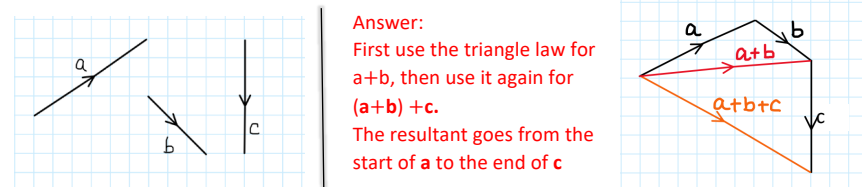
The law states that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

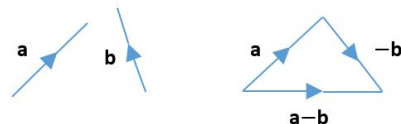
If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$ then, $\mathbf{a} + \mathbf{b} = \mathbf{c}$



Example 1: The diagram shows vectors a, b and c. Draw a diagram to illustrate the vector addition



Subtracting a vector is same as adding a negative vector $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$



Multiplying a vector:

You can multiply a vector by a scalar. If the number is $+ve$ the resultant vector has same direction but different length. If the number is $-ve$, the new vector has different length but opposite direction

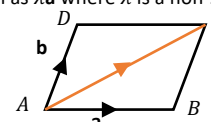


Any vector parallel to the vector a may be written as $\lambda \mathbf{a}$ where λ is a non-zero scalar

Parallelogram Law for vector addition:

For the parallelogram ABCD,

If $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AD} = \mathbf{b}$ then $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$



Example 2: Show that the vectors $6\mathbf{a} + 8\mathbf{b}$ and $9\mathbf{a} + 12\mathbf{b}$ are parallel.

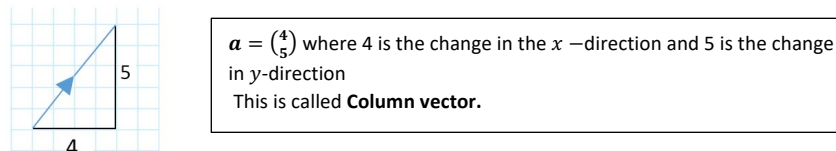
Try to write one of the vectors as $\lambda \times$ other vector

$$9\mathbf{a} + 12\mathbf{b} = \frac{3}{2}(6\mathbf{a} + 8\mathbf{b})$$

Since, $9\mathbf{a} + 12\mathbf{b}$ can be written as $\lambda \times (6\mathbf{a} + 8\mathbf{b})$ where $\lambda = \frac{3}{2}$, this implies that $6\mathbf{a} + 8\mathbf{b}$ and $9\mathbf{a} + 12\mathbf{b}$ are parallel.

Representing vector:

A vector represent change in position or displacement relative to the change in x- and y -axes



To multiply a column vector by a scalar multiply each component by the scalar: $\lambda \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \lambda p \\ \lambda q \end{pmatrix}$

To add 2 column vector add the x- component and the y-components: $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$

Example 3: Given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$, find the vector $4\mathbf{b} - \mathbf{a}$ in terms of i and j

Start by substituting vectors a and b in $4\mathbf{b} - \mathbf{a}$ and then simplify

$$4\mathbf{b} - \mathbf{a} = 4(4\mathbf{i} - \mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})$$

$$= (16\mathbf{i} - 4\mathbf{j}) - (2\mathbf{i} + 3\mathbf{j}) = (16 - 2)\mathbf{i} + (-4 - 3)\mathbf{j} = 14\mathbf{i} - 7\mathbf{j}$$

Hence, $4\mathbf{b} - \mathbf{a}$ in terms of i and j is $14\mathbf{i} - 7\mathbf{j}$

Example 4: Given that $\mathbf{a} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$ find $2\mathbf{a} + \mathbf{b} + \mathbf{c}$

Start by substituting vectors a, b and c in $2\mathbf{a} + \mathbf{b} + \mathbf{c}$

$$2\mathbf{a} + \mathbf{b} + \mathbf{c} = 2\begin{pmatrix} 9 \\ 7 \end{pmatrix} + \begin{pmatrix} 11 \\ -3 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \end{pmatrix} = \begin{pmatrix} 18 \\ 14 \end{pmatrix} + \begin{pmatrix} 11 \\ -3 \end{pmatrix} + \begin{pmatrix} -8 \\ -1 \end{pmatrix} = \begin{pmatrix} 21 \\ 10 \end{pmatrix}$$

Unit Vector:

A unit vector is a vector of length 1.

The unit vectors along the x- and y-axes are usually denoted by i and j respectively.

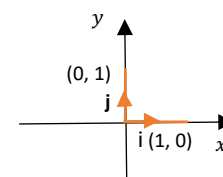
$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

You can write any two-dimensional vector in the form $\begin{pmatrix} p \\ q \end{pmatrix} = p\mathbf{i} + q\mathbf{j}$

In this topic you will learn to find a unit vector in the direction of a given vector

A unit vector in the direction of a is $\frac{\mathbf{a}}{|\mathbf{a}|}$

so if $|\mathbf{a}| = 5$, then a unit vector in the direction of a is $\frac{\mathbf{a}}{5}$



Magnitude and direction:

To calculate the magnitude of vector you can use Pythagoras theorem.

For the vector $\mathbf{a} = xi + yj = \begin{pmatrix} x \\ y \end{pmatrix}$ the magnitude of the vector is given by

$$|\mathbf{a}| = \sqrt{x^2 + y^2}$$

The two straight lines on either side of the vector is the notation for magnitude of a vector

Example 4: $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 4\mathbf{j}$. Find the exact value of $\mathbf{a} + \mathbf{b}$

Since we need to calculate exact value of $\mathbf{a} + \mathbf{b}$, first we will have to add the two vectors.

We can do that by representing the vectors in column vector form and then add

So, $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ implies $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 3-4 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

Use the formula for magnitude which is $|\mathbf{a}| = \sqrt{x^2 + y^2}$

$$\text{Hence, } |\mathbf{a} + \mathbf{b}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}$$

Keep your answer in surd form as you are asked to find exact value.

Example 5: For vector $\mathbf{a} = 5\mathbf{i} - 12\mathbf{j}$, find the unit vector in the same direction.

A unit vector in the direction of a is $\frac{\mathbf{a}}{|\mathbf{a}|}$

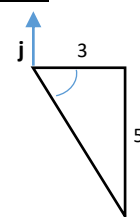
$$\text{Let us first calculate } |\mathbf{a}|, \text{ so } |\mathbf{a}| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

To find the unit vector in the direction of a first write the vector in column vector form and multiply by $\frac{1}{|\mathbf{a}|}$

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}}{13} = \frac{1}{13} \begin{pmatrix} 5 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} \\ -\frac{12}{13} \end{pmatrix}$$

Hence, the unit vector in the direction of a is $\begin{pmatrix} \frac{5}{13} \\ -\frac{12}{13} \end{pmatrix}$

Example 6: Find the angle that the vector $3\mathbf{i} - 5\mathbf{j}$ makes with j axis



The angle the vector $3\mathbf{i} - 5\mathbf{j}$ makes with j is $90^\circ + \text{angle } \theta$

Vector $3\mathbf{i} - 5\mathbf{j}$ on axes will be 3 units to the right on i axis and 5 units down on j axis

We can find angle θ by using trigonometric ratio $\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{5}{3}$

So, $\tan \theta = \frac{5}{3}$ taking inverse we get $\theta = \tan^{-1}(\frac{5}{3}) = 59^\circ$

So the angle $3\mathbf{i} - 5\mathbf{j}$ makes with j is $90^\circ + 59^\circ = 149^\circ$

Position Vector:

Position vector are the vectors that gives us the position of a point relative to fixed origin

In general, a point P with coordinates (p, q) has a position vector

$$\overrightarrow{OP} = p\mathbf{i} + q\mathbf{j} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Example 7: $\overrightarrow{OA} = 5\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$. Find the position vector of B

Start by drawing the vectors on the axes to find position vector of B. From the diagram position vector of B is vector \overrightarrow{OB} . It is easier to work with column vectors, so write the vectors

in column vector form

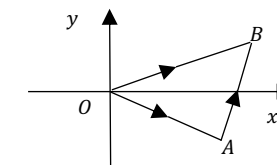
$$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} \text{ and } \overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Using triangle law of vector addition vector

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

$$\overrightarrow{OB} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5+3 \\ -2+4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$

$$\text{In i, j form } \overrightarrow{OB} = 8\mathbf{i} + 2\mathbf{j}$$



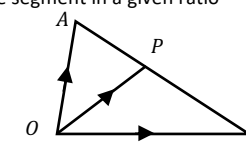
Solving geometric problem:

You can use vector to find the position vector of a point that divides a line segment in a given ratio

If the point P divides the line segment AB in the ratio $\lambda : \mu$, then

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu} \overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu} \overrightarrow{OB} - \overrightarrow{OA}$$



Example 8: In the diagram the points A and B have position vectors a and b respectively.

The point P divides AB in the ratio 1:2. Find the position vector of P

The position vector of P is vector \overrightarrow{OP} . You can refer to the figure above. To find \overrightarrow{OP} we can make use of the equation given above for point P dividing the line segment AB in the ratio $\lambda : \mu$. Here $\lambda = 1$ and $\mu = 2$ and substituting them in to the equation gives us,

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{3} \overrightarrow{AB}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{3} (\overrightarrow{OB} - \overrightarrow{OA})$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{3} \overrightarrow{OB} - \frac{1}{3} \overrightarrow{OA}$$

$$= \overrightarrow{OA} - \frac{1}{3} \overrightarrow{OA} + \frac{1}{3} \overrightarrow{OB} = \frac{2}{3} \overrightarrow{OA} + \frac{1}{3} \overrightarrow{OB} = \frac{2}{3} \mathbf{a} + \frac{1}{3} \mathbf{b}$$

$$\text{Hence, } \overrightarrow{OP} \text{ is } \frac{2}{3} \mathbf{a} + \frac{1}{3} \mathbf{b}$$

You will have to write \overrightarrow{AB} in terms of the position vectors of A and B in order to find \overrightarrow{OP}

If a and b are two non-parallel vectors and $p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b}$ then $p = r$ and $q = s$

Modelling with vectors

In mechanics, vectors have both magnitude and direction. For example velocity, displacement, and force.

Magnitude of a vector is a scalar quantity - it has size but no direction

Vector quantity	Its magnitude
velocity	speed
displacement	distance

Example 9: A girl walks 2km due east from fixed point O to A, and then 3km due south from A to B. Find: i. The position vector of B relative to O ii. the bearing of B from O

i) To find the position vector of B relative to O,

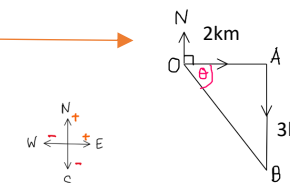
We will have to represent the girl's journey on a diagram

$$\overrightarrow{OB} = p\mathbf{i} + q\mathbf{j}$$

2km due east implies $p = 2$ and 3km due south

implies $q = -3$, you can refer to the diagram here

$$\text{So } \overrightarrow{OB} = (2\mathbf{i} - 3\mathbf{j}) \text{ km}$$



ii) To find bearing of B from O, you will have to calculate angle clockwise from North of O to

line segment OB which is $90^\circ + \text{angle } \theta$

Using $\tan \theta$ we can find angle θ

$$\tan \theta = \frac{3}{2}$$

$$\theta = \tan^{-1} \frac{3}{2} = 56.3^\circ$$

So the bearing of B from O is $90^\circ + 56.3^\circ = 146.3^\circ = 146^\circ$

Example 10: Find the distance moved by particle which travels for 10 seconds at velocity $(5\mathbf{i} - \mathbf{j}) \text{ ms}^{-1}$

You are familiar with the formula for distance, Distance = speed \times time

In order to calculate distance we need to find speed (i.e. magnitude of velocity)

and substitute speed and time = 10 seconds in the formula.

Use the magnitude formula to calculate speed

$$\text{Speed} = |5\mathbf{i} - \mathbf{j}| = \sqrt{5^2 + (-1)^2} = \sqrt{25 + 1} = \sqrt{26} = 5.099$$

$$\text{Hence, Distance} = 5.099 \times 10 = 51.0 \text{ cm}$$